

The Tutte polynomial of some matroids

Criel Merino*, Marcelino Ramírez-Ibañez[†]
 Guadalupe Rodríguez-Sánchez[‡]

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Abstract

The Tutte polynomial of a graph or a matroid, named after W. T. Tutte, has the important universal property that essentially any multiplicative graph or network invariant with a deletion and contraction reduction must be an evaluation of it. The deletion and contraction operations are natural reductions for many network models arising from a wide range of problems at the heart of computer science, engineering, optimization, physics, and biology. Even though the invariant is $\#P$ -hard to compute in general, there are many occasions when we face the task of computing the Tutte polynomial for some families of graphs or matroids. In this work we compile known formulas for the Tutte polynomial of some families of graphs and matroids. Also, we give brief explanations of the techniques that were used to find the formulas. Hopefully, this will be useful for researchers in Combinatorics and elsewhere.

*Instituto de Matemáticas, Universidad Nacional Autónoma de México, Area de la Investigación Científica, Circuito Exterior, C.U. Coyoacán 04510, México, D.F. México. e-mail: merino@matem.unam.mx. Supported by Conacyt of México Project 83977

[†]Escuela de Ciencias, Universidad Autónoma Benito Juárez de Oaxaca, Oaxaca, México. e-mail: marcelino@matem.unam.mx

[‡]Departamento de Ciencias Básicas Universidad Autónoma Metropolitana, Azcapotzalco, Av. San Pablo No. 180, Col. Reynosa Tamaulipas, Azcapotzalco 02200, México D.F. México. e-mail: rsmg@correo.azc.uam.mx

1 Introduction

Many times as researchers in Combinatorics we face the task of computing an evaluation of the Tutte polynomial of a family of graphs or matroids. Sometimes this is not an easy task, or at least time consuming. Later, not surprisingly, we find out that a formula was known for a class of graphs or matroids that contains our family. Here we survey some of the best known formulas for some interesting families of graphs and matroids. Our hope is for researchers to have a place to look for a Tutte polynomial before engaging in the search for the Tutte polynomial formula for the considered family.

We present along with the formulas, some explanation of the techniques used to compute them. This may also provide tools for computing the Tutte polynomials of new families of graphs or matroids. This survey can also be considered a companion to the book chapter Graph polynomials and their applications I: the Tutte polynomial by J. Ellis-Monaghan and C. Merino [25]. There, the authors give an introduction of the Tutte polynomial for a general audience of scientists, pointing out relevant relations between different areas of knowledge. But very few explicit calculations are made. Here we consider the practical side of computing the Tutte polynomial.

However, we are not presenting evaluations, that is an immense area of research for the Tutte polynomial, nor analysing the complexity of computing the invariant for the different families. For the former we strongly recommend the book of D.J.A. Welsh [47], for the latter we recommend S.D. Noble's book chapter [36]. There are many sources for the theory behind this important invariant. The most useful is definitively J. Oxley and T. Brylawski book chapter [16]. We already mentioned [25], which also surveys a variety of information about the Tutte polynomial of a graph, some of it new.

We also do not address the closely related problem of characterizing families of matroids by their Tutte polynomial, a problem which is generally known as Tutte uniqueness. For this there are also several articles, for example [9, 23, 38].

We assume knowledge of graph theory as in [24] and matroid theory as in [37]. Further details of many of the concepts treated here can be found in Welsh [47] and Oxley and Brylawski [16].

2 Definitions

Some of the richness of the Tutte polynomial is due to its numerous equivalent definitions; which is probably inherited from the vast number of equivalent definitions of the concept of matroid. In this chapter we revise three definitions and we put them to practice by computing the Tutte of some families of matroids, in particular uniform matroids.

2.1 The rank-nullity generating function definition

One of the simplest definitions, which is often the easiest way to prove properties of the Tutte polynomial, uses the notion of rank.

If $M = (E, r)$ is a matroid, where r is the rank-function of M , and $A \subseteq E$, we denote $r(E) - r(A)$ by $z(A)$ and $|A| - r(A)$ by $n(A)$, the latter is called the nullity of A .

Definition 2.1. *The Tutte polynomial of M , $T_M(x, y)$, is defined as follows:*

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}. \quad (1)$$

Duality

Recall that if $M = (E, r)$ is a matroid, then $M^* = (E, r^*)$ is its dual matroid, where $r^*(A) = |A| - r(E) + r(E \setminus A)$. Because $z_{M^*}(A) = n_M(E \setminus A)$ and $n_{M^*}(A) = z_M(E \setminus A)$, you get the following equality

$$T_M(x, y) = T_{M^*}(y, x). \quad (2)$$

This gives the first technique to compute a Tutte polynomial. If you know the Tutte polynomial of M then you know the Tutte polynomial of the dual matroid M^* .

Uniform matroids

Our first example is the family of uniform matroids $U_{r,n}$, where $0 \leq r \leq n$. Here the Tutte polynomial can be computed easily using (1) because all subsets of size $k \leq r$ are independent, so $n(\cdot)$ is zero; all subsets of size $k \geq r$

are spanning, so $z(\cdot)$ is zero and if a subset is independent and spanning, then it is a basis of the matroid.

$$T_{U_{r,n}}(x, y) = \sum_{i=0}^{r-1} \binom{n}{i} (x-1)^{r-i} + \binom{n}{r} + \sum_{i=r+1}^n \binom{n}{i} (y-1)^{i-r}. \quad (3)$$

Thus, for $U_{2,5}$ we get

$$\begin{aligned} T_{U_{2,5}}(x, y) &= (x-1)^2 + 5(x-1) + 10 + 10(y-1) + 5(y-1)^2 + (y-1)^3 \\ &= x^2 + 3x + 3y + 2y^2 + y^3. \end{aligned}$$

As $U_{3,5} = (U_{2,5})^*$, we get by using equation (2)

$$T_{U_{3,5}}(x, y) = x^3 + 2x^2 + 3x + 3y + y^2.$$

Matroid relaxation

Given a matroid $M = (E, r)$ with a subset $X \subseteq E$ that is both a circuit and a hyperplane, we can define a new matroid M' as the matroid with basis $\mathcal{B}(M') = \mathcal{B}(M) \cup \{X\}$. That M' is indeed a matroid is easy to check. For example, F_7^- is the unique relaxation of F_7 . The Tutte polynomial of M' can be computed easily from the Tutte polynomial of M by using equation (1).

$$T_{M'}(x, y) = T_M(x, y) - xy + x + y. \quad (4)$$

Matroid relaxation will be used extensively in the last section.

Sparse paving matroids

Our second example can be considered a generalization of uniform matroids but it is a much larger and richer family. A *paving matroid* $M = (E, r)$ is a matroid whose circuits all have size at least r . Uniform matroids are an example of paving matroids. Paving matroids are closed under minors and the set of excluded minors for the class consists of the matroid $U_{2,2} \oplus U_{0,1}$, see for example [33]. The interest about paving matroids goes back to 1976 when Dominic Welsh asked if most matroids were paving, see [37].

Sparse paving matroids were introduced in [28, 32]. A rank- r matroid M is sparse paving if M is paving and for every pair of circuits C_1 and C_2 of size r we have $|C_1 \triangle C_2| > 2$. For example, all uniform matroids are sparse paving matroids.

There is a simple characterization of paving matroids which are sparse in terms of the sizes of their hyperplanes. For a proof, see [33].

Theorem 2.2. *Let M be a paving matroid of rank $r \geq 1$. Then M is sparse paving if and only if all the hyperplanes of M have size r or $r - 1$.*

Note that we can say a little more, any circuit of size r is a hyperplane. Conversely, any proper subset of a hyperplane of size r is independent and so such a hyperplane must be a circuit. Thus, the circuits of size r are precisely the hyperplanes of size r .

Many invariants that are usually difficult to compute for a general matroid are easy for sparse paving matroids. For example, observe that if M is sparse paving, all subsets of size $k < r$ are independent, and all subsets of size $k > r$ are spanning. On the other hand the subsets of size r are either bases or circuit-hyperplanes. Thus, the Tutte polynomial of a rank- r sparse matroid M with n elements and λ circuit-hyperplanes is given by

$$T_M(x, y) = \sum_{i=0}^{r-1} \binom{n}{i} (x-1)^{r-i} + \binom{n}{r} + \lambda(xy-x-y) + \sum_{i=r+1}^n \binom{n}{i} (y-1)^{i-r}. \quad (5)$$

Clearly, given a r -rank paving matroid M with n elements, we can obtain the uniform matroid $U_{r,n}$ by a sequence of relaxations from M . If M has λ circuits-hyperplanes, equation (4) also gives (5).

Free extension

Another easy formula that we can obtained from the above definition involves the free extension, $M + e$, of a matroid $M = (E, r)$ by an element $e \notin E$, which consists of adding the element e to M as independently as possible without increasing the rank. Equivalently, the rank function of $M + e$ is given by the following equations: for X a subset of E ,

$$r_{M+e}(X) = r_M(X),$$

and

$$r_{M+e}(X \cup e) = \begin{cases} r_M(X) + 1, & \text{if } r_M(X) < r(M); \\ r(M), & \text{otherwise.} \end{cases}$$

Again, the Tutte polynomial of $M + e$ can be computed easily from the Tutte polynomial of M by using equation (1).

$$T_{M+e}(x, y) = \frac{x}{x-1} T_M(x, y) + \left(y - \frac{x}{x-1} \right) T_M(1, y). \quad (6)$$

Here the trick is to notice that $T_M(1, y) = \sum_X (y-1)^{|X|-r(X)}$, where the sum is over all subsets of E with $r(X) = r(E)$, i.e., spanning subsets of M . The presentation given here is from [10], but the formula can also be found in [15].

For example, the graphic matroid $M(K_4 \setminus e)$, has as free extension the matroid Q_6 . Because $M(K_4 \setminus e)$ is sparse paving, by using (5), we can compute its The Tutte polynomial and obtain

$$T_{K_4 \setminus e}(x, y) = x^3 + 2x^2 + x + 2xy + y + y^2.$$

Now, by using (6), we get the Tutte polynomial of the free extension.

$$T_{Q_6} = x^3 + 3x^2 + 4x + 2xy + 4y + 3y^2 + y^3.$$

This result can be check by using (5), as Q_6 is also sparse paving.

2.2 Deletion and contraction

In the second (equivalent) definition of the Tutte polynomial we use a linear recursion relation given by deleting and contracting elements that are neither loops nor coloops. This is by far the most used method to compute the Tutte polynomial.

Definition 2.3. *If M is a matroid, and e is an element that is neither a loop nor a coloop, then*

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y). \quad (7)$$

If e is a coloop, then

$$T_M(x, y) = x T_{M \setminus e}(x, y). \quad (8)$$

And if e is a loop, then

$$T_M(x, y) = y T_{M/e}(x, y). \quad (9)$$

The proof that Definition 2.1 and 2.3 are equivalent can be found in [16]. From this it is clear that you just need the Tutte polynomial of the matroid without loops and coloops. Also, if you know the Tutte polynomial of $M \setminus e$ and M/e , then you know the Tutte polynomial of the matroid M . This way of computing the Tutte polynomial leads naturally to linear recursions. We present two examples where these linear recursions give formulas.

The cycle C_n

As a first example of this subsection we consider the graphic matroid $M(C_n)$. Here, by deleting an edge e from C_n we obtained a path whose Tutte polynomial is x^{n-1} , while if we contract e we get a smaller cycle C_{n-1} . Thus, as the Tutte polynomial of the 2-cycle is $x + y$, we obtain

$$T_{C_n}(x, y) = \sum_{i=1}^{n-1} x^i + y. \quad (10)$$

By duality, the Tutte polynomial of the graph C_n^* with two vertices and n parallel edges between them is

$$T_{C_n^*}(x, y) = \sum_{i=1}^{n-1} y^i + x. \quad (11)$$

Parallel and series classes

As an almost trivial application of deletion-contraction we look into the common case when you have a graph or a matroid with parallel elements. Let us define a *parallel class* in a matroid M as maximal subset X of $E(M)$ such that any two distinct members of X are parallel and no member of X is a loop, see [37]. Then the following result is well known and has been found many times, see [22, 26].

Lemma 2.4. *Let X be a parallel class of a matroid M with $|X| = p + 1$. If X is not a cocircuit, then*

$$T_M(x, y) = T_{M \setminus X}(x, y) + (y^p + y^{p-1} + \dots + 1)T_{M/X}(x, y). \quad (12)$$

The proof is by induction on p with the case $p = 0$ being (7). The rest of the proof follows easily from the fact that each loop introduces a multiplicative factor of y in the Tutte polynomial.

A *series class* in M is just a parallel class in M^* and by duality we get

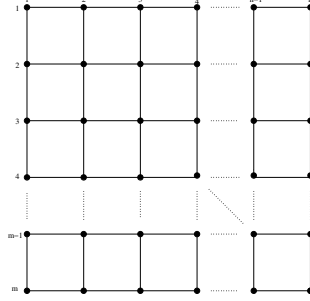


Figure 1: The grid graph $L_{m,n}$.

Lemma 2.5. *Let X be an series class of a matroid M with $|X| = p + 1$. If X is not a circuit, then*

$$T_M(x, y) = (x^p + x^{p-1} + \dots + 1)T_{M \setminus X}(x, y) + T_{M/X}(x, y). \quad (13)$$

The rectangular lattice $L_{m,n}$

The rectangular lattice, that we will define in a moment, is our first example where computing its Tutte polynomial is really far from trivial and no complete answer is known. The interest resides probably in the importance of computing the Potts partition function, which is equivalent to the Tutte polynomial, of the square lattice. However, even a complete resolution of this problem will be just a small step towards the resolution of the really important problem of computing the Tutte polynomial of the cubic lattice.

Let m and n be integers, $m, n > 1$. The *grid* or *rectangular lattice* $L_{m,n}$ is a connected planar graph with mn vertices, such that its faces are squares, except one face that it is a polygon with $2(m+n) - 4$ edges. The grid graph $L_{m,n}$ can be represented as in Figure 1

The vertices of $L_{m,n}$ are denoted as ordered pairs, such that the vertex in the intersection of the row i and the column j is denoted by ij .

As example, we show a recursive formula for the Tutte polynomials of grid graphs $L_{m,n}$ when $m = 2$. For $m > 3$, the formulas are very complicated to use in practical calculations.

The graphs $L_{2,1}$, $L_{2,2}$ and $L_{2,3}$ are shown in the Figure 2, with the labels that correspond to their vertices.

For every positive integer n . The graph $Q_{2,n}$ is defined from the graph

$L_{2,n}$ by contraction of the edge $(11, 21)$ of $L_{2,n}$. In particular, the graph $Q_{2,1}$ is one isolated vertex, see Figure 2



Figure 2: The first three terms are from the sequence $\{L_{2,n}\}$ and the last three are from $\{Q_{2,n}\}$.

The initial conditions to construct the recurrence relations for the Tutte polynomial of the grid graphs $L_{2,n}$ are:

- i. $T_{Q_{2,1}}(x, y) = 1$
- ii. $T_{L_{2,1}}(x, y) = x$
- iii. $T_{L_{2,2}}(x, y) = x^3 + x^2 + x + y$

The third condition is true because the graph $L_{2,2}$ is isomorphic to the cycle C_4 , so we can use Equation (10). The formula for the recurrence relation to calculate the Tutte polynomial of the graphs $L_{2,n}$ is:

$$T_{L_{2,n}}(x, y) = (x^2 + x + 1)T_{L_{2,n-1}}(x, y) + yT_{Q_{2,n-1}}(x, y) \quad (14)$$

$$T_{Q_{2,n}}(x, y) = (x + 1)T_{L_{2,n-1}}(x, y) + yT_{Q_{2,n-1}}(x, y). \quad (15)$$

From this, we get a linear recurrence of order two,

$$T_{L_{2,n}} - (x^2 + x + 1 + y)T_{L_{2,n-1}} + x^2yT_{L_{2,n-2}} = 0,$$

that is easy to solve. Thus, the general formula is

$$T_{L_{2,n}}(x, y) = \frac{a_1\lambda_1 + a_2}{\lambda_1 - \lambda_2}\lambda_1^{n-2} + \frac{a_1\lambda_2 + a_2}{\lambda_2 - \lambda_1}\lambda_2^{n-2}, \quad (16)$$

for $n \geq 2$, where $a_1 = y + x + x^2 + x^3$, $a_2 = -yx^3$ and λ_1 and λ_2 are

$$\frac{1}{2} \left[(1 + y + x + x^2) \pm ((y^2 + 2y(1 + x - x^2) + (1 + x + x^2)^2)^{1/2} \right]$$

A *recursive family of graphs* G_m is a sequence of graphs with the property that the Tutte polynomials $T_{G_m}(x, y)$ satisfy a linear homogeneous recursion relation in which the coefficients are polynomials in x and y with integral coefficients, independent of m , see [5]. So, the sequence $\{L_{2,n}\}$ is such a family and in general, $\{L_{k,n}\}$, for a fixed k , is a recursive family.

Graphic and representable matroids

Probably the most common class of matroids, when evaluations of the Tutte polynomial are considered, are graphic matroids. If M is the graphic matroid of a graph G , then deleting an element e from M amounts to deleting the corresponding edge from G , and contracting e to contracting the edge. This is easy to do in a small graph and by using Definition 2.3 you can compute the Tutte polynomial. An example is shown in Figure 3

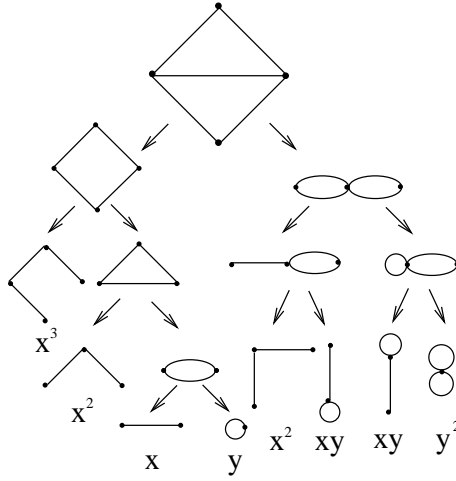


Figure 3: An example of computing the Tutte polynomial of a graphic matroid using deletion and contraction

As we mentioned computing the Tutte polynomial is not in general computationally tractable. However, for graphic matroids there are some resources to compute it for reasonably sized graphs of about 100 edges. These include Sekine, Imai, and Tani [40], which provides an algorithm to implement the recursive Definition 2.3. Common computer algebra systems such as Maple and Mathematica will compute the Tutte polynomial for very small graphs, and there are also some implementations freely available on the Web, such as <http://ada.fciencias.unam.mx/~rconde/tulic/> by R. Conde or <http://homepages.mcs.vuw.ac.nz/~djp/tutte/> by G. Haggard and D. Pearce.

Similarly, given a matroid $M = (E, r)$, representable over a field \mathbb{F} , you can compute its Tutte polynomial using deletion and contraction. For any

element $e \in E$, the matroids $M \setminus e$ and M/e are easily computed from the representation of M , see [37]. This can be automatized and there are already computer programs that compute the Tutte polynomial of a representable matroid. For our calculations we have used the one given by Michael Barany, for more information about this program and how to use it, see [2]. Some of the calculations made in the Section 4 were computed using this program.

2.3 Internal and external activity

Being a polynomial, it is natural to ask for the coefficients of the Tutte polynomial. Tutte's original definition gives its homonymous polynomial in terms of its coefficients by means of a combinatorial interpretation, but before we give the third definition of the Tutte polynomial, we introduce the relevant notions.

Let us fix an ordering \prec on the elements of M , say $E = \{e_1, \dots, e_m\}$, where $e_i \prec e_j$ if $i < j$. Given a fixed basis S , an element e is called *internally active* if $e \in S$ and it is the smallest edge with respect to \prec in the only cocircuit disjoint from $S \setminus \{e\}$. Dually, an element f is *externally active* if $f \notin S$ and it is the smallest element in the only circuit contained in $S \cup \{f\}$. We define t_{ij} to be the number of bases with i internally active elements and j externally active elements. In [44, 45, 46] Tutte defined T_M using these concepts. A proof of the equivalence with Definition 2.1 can be found in [6].

Definition 2.6. *If $M = (E, r)$ is a matroid with a total order on its ground set, then*

$$T_M(x, y) = \sum_{i,j} t_{ij} x^i y^j. \quad (17)$$

In particular, the coefficients t_{ij} are independent of the total order used on the ground set.

Uniform matroids again

Our first use of (17) is to get a slightly different expression for the Tutte polynomial of uniform matroids. This time we get

$$T_{U_{r,n}}(x, y) = \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} y^j + \sum_{i=1}^r \binom{n-i-1}{n-r-1} x^i, \quad (18)$$

when $0 < r < n$, while $T_{U_{n,n}}(x, y) = x^n$ and $T_{U_{0,n}}(x, y) = y^n$. This can also be established by expanding (3).

Paving matroids

Paving matroids were defined above. The importance of paving matroids is its abundance, meaning, most matroids of up to 9 elements are paving. This was checked in [32] and it has been conjectured in [31] that this is indeed true for all matroids, that is, when n is large, the probability that you choose a paving matroid uniformly at random among all matroids with up to n elements is approaching 1.

Now, in order to get a formula for the Tutte polynomial of paving matroids, we need the following definition from [37]. Given integers $k > 1$ and $m > 0$, a collection $\mathcal{T} = \{T_1, \dots, T_k\}$ of subsets of a set E , such that each member of \mathcal{T} has at least m elements and each m -element subset of E is contained in a unique member of \mathcal{T} , is called an m -partition of E . The elements of the partition are called *blocks*. The following proposition is also from [37].

Proposition 2.7. *If \mathcal{T} is an m -partition of E , then \mathcal{T} is the set of hyperplanes of a paving matroid of rank $m + 1$ on E . Moreover, for $r \geq 2$, the set of hyperplanes of every rank- r paving matroid on E is an $(r - 1)$ -partition of E .*

Brylawski gives the following proposition in [13], but we mentioned that his proof does not use activities.

Proposition 2.8. *Let M be a rank- r matroid with n elements and Tutte polynomial $\sum_{i,j} t_{ij} x^i y^j$. Then, M is paving iff $t_{ij} = 0$ for all $(i, j) \geq (2, 1)$. In addition if the $(r - 1)$ -partition of E has b_k blocks of cardinality k , for $k = r - 1, \dots, n$, then*

$$\begin{aligned} t_{i0} &= \binom{n-i-1}{r-i} \text{ for all } i \geq 2; \\ t_{10} &= \sum_{k=0}^{\infty} \binom{r-2+k}{r-2} b_{k+r-1} + \binom{n-2}{r-1} - \binom{n}{r-1}; \end{aligned}$$

and for all $j > 0$,

$$t_{1j} = \sum_{k=0}^{\infty} \binom{r-2+k}{r-2} b_{k+j+r-1}, \text{ and}$$

$$t_{0j} = \binom{n-j-1}{r-1} - \sum_{k=0}^{\infty} \binom{r-1+k}{r-1} b_{k+j+r-1}.$$

Note that when M is the uniform matroid $U_{r,n}$, M is a paving matroid with $(r-1)$ -partition all subsets of size $r-1$. Thus the above formula give us equation (18).

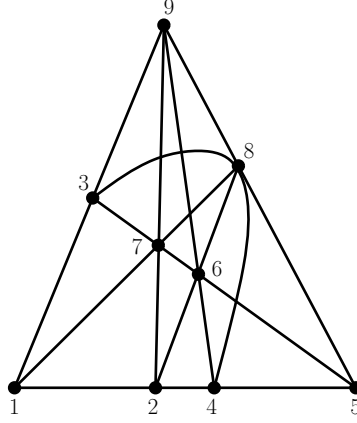


Figure 4: The matroid R_9

As a example consider the matroid R_9 with geometric representation given in Figure 4. This matroid is paving but not sparse paving. The blocks correspond to lines in the representation. There are 7 blocks of size 3, 2 blocks of size 4 and 3 blocks of size 2, corresponding to trivial lines that do not appear in Figure 4. Using the above formulas we get that the Tutte polynomial of R_9 is

$$T_{R_9}(x, y) = x^3 + 6x^2 + 8x + 11xy + 2xy^2 + 8y + 13y^2 + 10y^3 + 6y^4 + 3y^5 + y^6.$$

Catalan matroids

Our final example in this subsection is a family of matroids that although simple, it has a surprisingly natural combinatorial definition together with a nice interpretation of the internal and external activity.

Consider an alphabet constituted by the letters $\{N, E\}$. The *length* of a word $w = w_1 w_2 \dots w_n$ is n , the number of letters in w . Every word can be associated with a path on the plane \mathbb{Z}^2 , with a initial point $A = (m_1, n_1)$ and a final point $B = (m_2, n_2)$, $m_1 < m_2$ and $n_1 < n_2$. The letter N is identified with a north step and the letter E with a east step, such that the first step of $A = (0, 0)$ to B will be $N = (0, 1)$ or $E = (1, 0)$.

Let A and B be a couple of fixed points of \mathbb{Z}^2 . A *lattice path* is a path in \mathbb{Z}^2 from A to B using only steps N or E . The lattice paths from A to B have m steps E and n steps N . Let P and Q be two lattice paths, with initial points (x_0, y_P) and (x_0, y_Q) of P and Q respectively. If $y_P < y_Q$ for every x_0 in $[m_1, m_2]$; then the set of lattice paths in this work are lattice paths from A to B bounded between P and Q , that we called PQ -bounded lattice paths.

The lattice paths P and Q can be substituted by lines. If P is the line $y = 0$, Q the line $y = x$ and $m = n$, then the number of PQ -bounded lattice paths from $(0, 0)$ to (n, n) is the n -th Catalan number:

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Denote by $[n]$ the set $\{1, 2, \dots, n\}$. Consider the word $w = w_1, w_2, \dots, w_{m+n}$ that corresponds to a PQ -bounded lattice path. We can associate to w a subset X_w of $[m+n]$ by defining that $i \in X_w$ if $w_i = N$. X_w is the support set of the steps N in w . The family of support sets X_w of PQ -bounded lattice paths are the set of bases of a transversal matroid, denoted $M[P, Q]$ with ground set $[m+n]$, see [10].

If the bounds of the lattice paths are the lines $y = 0$ and $y = x$, the matroid is denoted by M_n and is named *Catalan matroid*. Note that M_n has a loop that corresponds to the label 1 and a coloop that corresponds to the label $2n$, and that M_n is autodual.

The fundamental result to compute the Tutte polynomial of Catalan matroids is the following, see [10], where here, for a basis B , we denote by $i(B)$ the number of internally active elements and by $e(B)$ the externally active elements.

Proposition 2.9. *Let $B \in \mathcal{B}$ be a basis of $M[P, Q]$ and let w_B be the PQ -bounded lattice path associated with B . Then $i(B)$ is the number of times w_B meets the upper path Q in a north step and $e(B)$ is the number of times w_B meets the lower path P in an east step.*

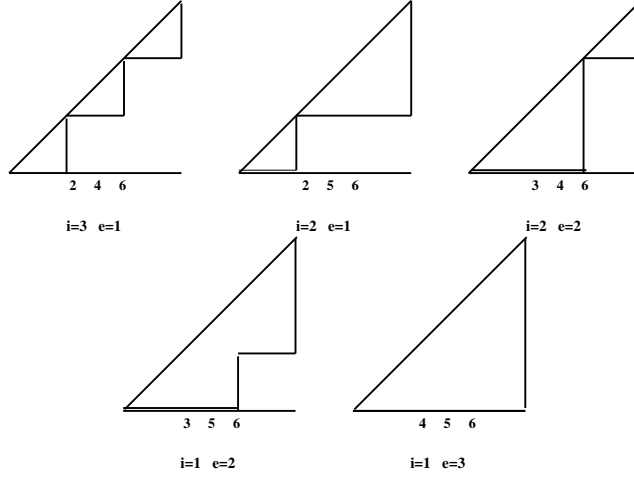


Figure 5: Bases of M_3 with their internal and external activity.

Then the Tutte polynomial of the Catalan matroids M_n for $n > 1$, is:

$$T_{M_n}(x, y) = \sum_{i, j \geq 0} \frac{i + j - 2}{n - 1} \binom{2n - i - j - 1}{n - i - j + 1} x^i y^j. \quad (19)$$

Note that the coefficient of $x^i y^j$ in the Tutte polynomial of the matroid M_n depends only on n and the sum $i + j$.

In the Figure 5 are shown the bases of M_3 with its internally and external activities. Thus the Tutte polynomial of M_3 is

$$T_{M_3}(x, y) = x^3 y + x^2 y + x^2 y^2 + x y^2 + x y^3.$$

3 Techniques

We move now to present three techniques that are sometimes useful to compute the Tutte polynomial, when the initial trial with the above definitions are not successful.

3.1 Equivalent polynomials: The coboundary polynomial

For some families of matroids and graphs it is easier to compute polynomials that are equivalent to the Tutte polynomial. One such polynomial is Crapo's coboundary polynomial, see also [14].

$$\bar{\chi}_M(\lambda, t) = \sum_{X \in \mathcal{F}_M} t^{|X|} \chi_{M/X}(\lambda) \quad (20)$$

where \mathcal{F}_M is the set of flats of M and $\chi_N(\lambda)$ is the characteristic polynomial of the matroid N . The characteristic polynomial of a matroid M is defined by

$$\chi_N(\lambda) = \sum_{A \subseteq E} (-1)^{|X|} \lambda^{z(A)}. \quad (21)$$

The Tutte and the coboundary polynomial are related as follows

$$T_M(x, y) = \frac{1}{(y-1)^{r(E)}} \bar{\chi}_M((x-1)(y-1), y). \quad (22)$$

$$\bar{\chi}_M(\lambda, t) = (t-1)^{r(E)} T_M\left(\frac{\lambda+t-1}{t-1}, t\right). \quad (23)$$

When M is a graphic matroid $M(G)$ of a connected graph G , the characteristic polynomial is equivalent to the chromatic polynomial, that we denote χ_G .

$$\chi_G(\lambda) = \lambda \chi_{M(G)}(\lambda).$$

And also, in this case, the coboundary is the bad-colouring polynomial. The *bad colouring polynomial* is the generating function

$$B_G(\lambda, t) = \sum_j b_j(G; \lambda) t^j,$$

where $b_j(G; \lambda)$ is the number of λ -colourings of G with exactly j bad edges. Note that when we set $t = 0$ we obtain the chromatic polynomial. Then, the polynomials are related as follows:

$$B(G; \lambda, t) = \lambda \bar{\chi}_{M(G)}(\lambda, t). \quad (24)$$

The q -cone

We base this section on the work of J. E. Bonin and H. Qin, see [11].

Definition 3.1. *Let M be a rank- r simple matroid representable over $GF(q)$. A matroid N is a q -cone of M with base S and apex a if*

1. *the restriction $PG(r, q)|_S$ of $PG(r, q)$ to the subset S is isomorphic to M ,*
2. *The point $a \in PG(r, q)$ is not contained in the closure of S , $Cl_{PG(r, q)}(S)$, in $PG(r, q)$, and*
3. *The matroid N is the restriction of $PG(r, q)$ to the set $\bigcup_{s \in S} Cl_{PG(r, q)}(\{a, s\})$.*

That is, one represents M as a set S of points in $PG(r, q)$ and constructs N by restricting $PG(r, q)$ to the set of points on the lines joining the points of S to the fixed point a outside the hyperplane of $PG(r, q)$ spanned by S . The basic result is by Kung who proved in [29].

Theorem 3.2. *For every q -cone N of a rank- r simple matroid M , we have*

$$\chi_N(\lambda) = (\lambda - 1)q^r \chi_M(\lambda/q).$$

To extend this result to $\bar{\chi}$ the authors in [11] identify the flats F of N and the contractions N/F of N by these flats. They prove the following formula

$$\bar{\chi}_N(\lambda, t) = t\bar{\chi}_M(\lambda, t^q) + q^r(\lambda - 1)\bar{\chi}_M(\lambda/q, t). \quad (25)$$

By using equation (22) we get a formula for the Tutte polynomial of N in terms of the Tutte polynomial of M .

Theorem 3.3. *If M is a rank- r matroid representable over $GF(q)$ and N is a q -cone of M , then*

$$T_N(x, y) = \frac{y(y^q - 1)^r}{(y - 1)^{r+1}} T_M\left(\frac{(x - 1)(y - 1)}{y^q - 1} + 1, y^q\right) + \frac{q^r(xy - x - y)}{y - 1} T_M\left(\frac{x + 1}{q} + 1, y\right).$$

Thus, for example, the 2-cone of the 3-point line is the matroid $PG(2, 2)$. The Tutte polynomial of the former matroid is just $x^2 + x + y$, then by using the above formula we get the Tutte polynomial of $PG(2, 2)$ to be

$$x^3 + 4x^2 + 3x + 7xy + 3y + 6y^2 + 3y^3 + y^4.$$

As the q -cone of $PG(r - 1, q)$ is $PG(r, q)$, this method can be used to compute the Tutte polynomial of any $PG(r, q)$, but we do this later using the coboundary polynomial directly.

Complete graphs

Given the apparent simplicity of many of the formulas for invariants in complete graphs, like the number of spanning trees or acyclic orientations, it is not so straightforward to compute the whole Tutte polynomial of complete graphs; many researches, however, have tried with different amounts of success. The amount of frustration after failing to compute this seemingly innocuous invariant of many of our colleagues was our original motivation for writing this survey paper.

Using the exponential formula, see Stanley[42], we can give an exponential generating function for the Tutte polynomial of the complete graphs. Let us denote the vertices of K_n by V and by B_n its bad-colouring. To compute B_n , observe that any λ -colouring partitions the vertices V into λ color classes of subsets V_i of vertices each of cardinality n_i , for $i = 1 \dots \lambda$. So, we have that $n_1 + \dots + n_\lambda = n$. The number of bad edges with both ends in the set V_i is $t^{\binom{n_i}{2}}$. Thus, by the exponential formula we get the following formula

$$\left(\sum_{n \in \mathbb{N}} t^{\binom{n}{2}} \frac{u^n}{n!} \right)^\lambda = 1 + \sum_{n \geq 1} B_n(\lambda, t) \frac{u^n}{n!}. \quad (26)$$

Now, we can use equation (24) to get

$$\left(\sum_{n \in \mathbb{N}} t^{\binom{n}{2}} \frac{u^n}{n!} \right)^\lambda = 1 + \lambda \sum_{n \geq 1} \bar{X}_n(\lambda, t) \frac{u^n}{n!}. \quad (27)$$

Let $T_n(x, y)$ be the Tutte polynomial of K_n . Tutte in [46] and Welsh in [48] give the following exponential generating function for $T_n(x, y)$ that follows from the previous equation and equation (22).

$$\left(\sum_{n \geq 0} y^{\binom{n}{2}} \frac{u^n}{n!} \right)^{(x-1)(y-1)} = 1 + (x-1) \sum_{n \geq 1} (y-1)^n T_n(x, y) \frac{u^n}{n!}. \quad (28)$$

Even that the previous formulas seem difficult to handle, equation (27) it is quite easy to use in Maple or Mathematica. For example, the following program in Maple can compute $T_{30}(x, y)$ in no time.

```
Coboundary:= proc(n,q,v) local i,x,g;
               g:=x->(add(v^(i*(i-1)/2)*x^i/i!,i=0..n))^q;
```

```

      simplify(eval(diff(g(x),x\ $n),x=0)/q);end proc;
T:= proc(n,x,y)
  simplify( (1/((y-1)^(n-1)))*subs({q=(x-1)*(y-1),v=y},
    Coboundary(n,q,v)) ) end proc;

```

By computing $T_5(x, y)$ we obtain

$$T_5(x, y) = y^6 + 4y^5 + x^4 + 5xy^3 + 10y^4 + 6x^3 + 10x^2y + 15xy^2 + 15y^3 + 11x^2 + 20xy + 15y^2 + 6x + 6y.$$

Complete bipartite graphs

It is just natural to use this technique for complete bipartite graphs $K_{n,m}$ with similar results as above. This time let the vertex set be $V_1 \cup V_2$ and denote by $B_{n,m}$ its bad-colouring polynomial. To compute $B_{n,m}$, observe that any λ -colouring partitioned the vertices in subsets $V_1^{(i)} \cup V_2^{(i)}$ each of cardinality $n_i m_i$. The number of the bad edges with both ends colour i is $n_i m_i$. Thus, by the exponential formula we get the following.

$$\begin{aligned}
\left(\sum_{(n,m) \in \mathbb{N}^2} t^{nm} \frac{u^n}{n!} \frac{v^m}{m!} \right)^\lambda &= 1 + \sum_{\substack{(n,m) \in \mathbb{N}^2 \\ (n,m) \neq (0,0)}} B_{n,m}(\lambda, t) \frac{u^n}{n!} \frac{v^m}{m!} \\
&= 1 + \lambda \sum_{\substack{(n,m) \in \mathbb{N}^2 \\ (n,m) \neq (0,0)}} \bar{\chi}_{n,m}(\lambda, t) \frac{u^n}{n!} \frac{v^m}{m!}.
\end{aligned} \tag{29}$$

Thus, a formula for the Tutte polynomial of the bipartite complete graph can be found in a similar way as before. Let $T_{n,m}(x, y)$ be the Tutte polynomial of $K_{n,m}$. The following formula is from Stanley's book [42], see also [30]

$$\begin{aligned}
\left(\sum_{(n,m) \in \mathbb{N}^2} y^{nm} \frac{u^n}{n!} \frac{v^m}{m!} \right)^{(x-1)(y-1)} &= 1 + (x-1) \\
&\quad \sum_{\substack{(n,m) \in \mathbb{N}^2 \\ (n,m) \neq (0,0)}} (y-1)^{nm} T_{n,m}(x, y) \frac{u^n}{n!} \frac{v^m}{m!}.
\end{aligned} \tag{30}$$

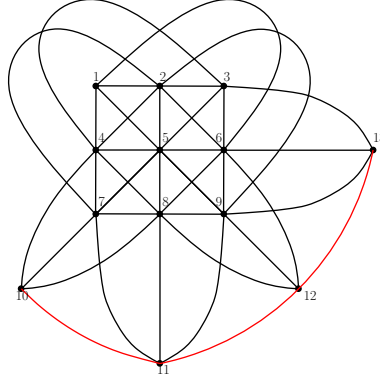


Figure 6: The projective plane $PG(2, 3)$ with a marked hyperplane

As before, equation (29) is quite easy to use in Maple to compute $T_{n,m}(x, y)$ for small values of n and m . In this way we get the following.

$$T_{3,3}(x, y) = x^5 + 4x^4 + 10x^3 + 9x^2y + 11x^2 + 6xy^2 + 15xy + 5x + y^4 + 5y^3 + 9y^2 + 5y. \quad (31)$$

Projective geometries and affine geometries

The role played by complete graphs in graphic matroids is taken by projective geometries in representable matroids. Herein lies the importance of projective geometries. Even though a formula for their Tutte polynomial has been known for a while, and has been discovered at least twice, not much work has been done in the actual combinatorial interpretations for the value of the different evaluations of the Tutte polynomial.

For this part we follow Mphako [34], see also [3]. For all non-negative integers m and k we define the *Gaussian coefficients* as

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}. \quad (32)$$

Note that $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1$ since by convention an empty product is 1 and $\begin{bmatrix} 0 \\ k \end{bmatrix}_q = 0$.

Let $PG(r - 1, q)$ be the $(r - 1)$ dimensional projective geometry over $GF(q)$. Then, as a matroid, it has rank r and $\begin{bmatrix} r \\ 1 \end{bmatrix}_q$ elements. Also, every rank- k flat X is isomorphic to $PG(k - 1, q)$ and the simplification of M/X

is isomorphic to $PG(r - k - 1, q)$. The number of rank- k flats is $\begin{bmatrix} r \\ k \end{bmatrix}_q$. The characteristic polynomial of $PG(r - 1, q)$ is known to be, see [16],

$$\chi_{PG(r-1,q)}(\lambda) = \prod_{i=0}^{r-1} (\lambda - q^i)$$

Thus, using equation (20) we get

$$\bar{\chi}_{PG(r-1,q)}(\lambda, t) = \sum_{k=0}^r t^{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \begin{bmatrix} r \\ k \end{bmatrix}_q \prod_{i=0}^{r-k-1} (\lambda - q^i). \quad (33)$$

The matroid $PG(2, 3)$, has a geometric representation given in Figure 6, and it is isomorphic to the unique Steiner system $S(2, 4, 13)$. The matroid is paving but not sparse paving and has the following representation over $GF(3)$.

$$\begin{array}{c|cccccccccccccc} 1 & 2 & 4 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \hline \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \end{array}.$$

To compute its Tutte polynomial we could use the program in [2] or use the above formula to get

$$\bar{\chi}_{PG(2,3)}(\lambda, t) = (\lambda - 1)(\lambda - 3)(\lambda - 9) + 13t(\lambda - 1)(\lambda - 3) + 13t^4(\lambda - 1) + t^{13}.$$

Finally the Tutte polynomial of $PG(2, 3)$ is obtained from the coboundary by a substitution $\lambda = (x - 1)(y - 1)$ and $t = y$ and by multiplying by the factor $1/(y - 1)^3$

$$\begin{aligned} T_{PG(2,3)}(x, y) &= x^3 + 10x^2 + 13xy^2 + 26xy + 16x + 16y + 32y^2 + 36y^3 \\ &\quad + 28y^4 + 21y^5 + 15y^6 + 10y^7 + 6y^8 + 3y^9 + y^{10}. \end{aligned} \quad (34)$$

Every time we have a projective geometry $PG(r - 1, q)$ we can get an *affine geometry*, $AG(r - 1, q)$, simply by deleting all the points in a hyperplane of $PG(r - 1, q)$. For example, by deleting all the red points from $PG(2, 3)$ in Figure 6 we get $AG(2, 3)$ with geometric representation in Figure 7.

The situation to compute the Tutte polynomial is very similar for the r -dimensional affine geometry over $GF(q)$, $AG(r, q)$. It has rank $r + 1$ and

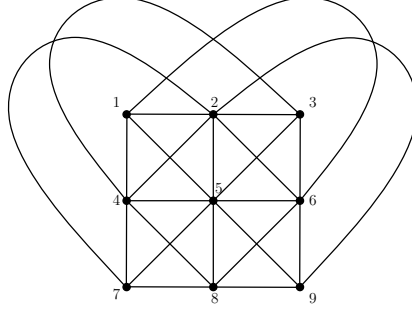


Figure 7: Affine plane $AG(2, 3)$

q^r elements. Any flat of X rank $k + 1$ is isomorphic to $AG(k, q)$ and the simplification of $AG(r, q)/X$ is isomorphic to $PG(r - k - 1, q)$. The number of rank- k flats is $q^{r-k} \begin{bmatrix} r \\ k \end{bmatrix}_q$. The characteristic polynomial of $AG(r, q)$ is known to be, see [16],

$$\chi_{AG(r,q)}(\lambda) = (\lambda - 1) \sum_{k=0}^r (-1)^k \lambda^{r-k} \prod_{i=0}^{k-1} (q^{r-i})$$

Now, using the result of Mphako we can compute the coboundary polynomial of $AG(r, q)$.

$$\bar{\chi}_{AG(r,q)}(\lambda, t) = \chi_{AG(r,q)}(\lambda) + \sum_{k=0}^r t^{q^k} q^{r-k} \begin{bmatrix} r \\ k \end{bmatrix}_q \prod_{i=0}^{r-k-1} (\lambda - q^i). \quad (35)$$

The affine plane $AG(2, 3)$ is isomorphic to the unique Steiner triple system $S(2, 3, 9)$, as every line contains exactly 3 points and every pair of points is in exactly one line. $S(2, 3, 9)$ is a sparse paving matroid with 12 circuit-hyperplanes so by using equation (5) we could compute its Tutte polynomial. However, we use the above formula to obtain the same result.

$$\bar{\chi}_{AG(2,3)}(\lambda, t) = (\lambda - 1)(\lambda^2 - 8\lambda + 16) + 9t(\lambda - 1)(\lambda - 3) + 12t^3(\lambda - 1) + t^9.$$

Again, the Tutte polynomial of $AG(2, 3)$ is obtained from the coboundary by a substitution $\lambda = (x - 1)(y - 1)$ and $t = y$ and by multiplying by the factor $1/(y - 1)^3$

$$T_{AG(2,3)}(x, y) = x^3 + 6x^2 + 12xy + 9x + 9y + 15y^2 + 10y^3 + 6y^4 + 3y^5 + y^6.$$

3.2 Transfer-matrix method

Using formula (1) quickly becomes prohibitive as the number of states grow exponentially with the size of the matroid. You can get around this problem when you have a family of graphs that are constructed using a simple graph that you repeat in a path-like fashion; the bookkeeping of the contribution of each state can be done with a matrix, the update can be done by matrix multiplication after the graph grows a little more. This is the essence of our second method.

The theoretical background of the transfer-matrix method, taken from [41], is described below.

A *directed graph* or *digraph* \vec{G} is a triple (V, E, ϕ) , where $V = \{v_1, \dots, v_p\}$ is a set of vertices, E is a finite set of *directed edges* or *arcs*, and ϕ is a map from E to $V \times V$. If $\phi(e) = (u, v)$, then e is called an edge from u to v , with *initial* vertex u and *final* vertex v . A *directed walk* Γ in \vec{G} of *length* n from u to v is a sequence e_1, \dots, e_n of n edges such that the final vertex of e_i is the initial vertex of e_{i+1} , for $1 \leq i \leq n-1$.

Now let $w : E \rightarrow R$ be a *weight function* on E with values in some commutative ring R . If $\Gamma = e_1, \dots, e_n$ is a walk, then the weight of Γ is defined by $w(\Gamma) = w(e_1) \cdots w(e_n)$. For $1 \leq i, j \leq p$ and $n \in \mathbb{N}$, we define

$$A_{i,j}(n) = \sum_{\Gamma} w(\Gamma),$$

where the sum is over all walks Γ in \vec{G} of length n from v_i to v_j . In particular, $A_{i,j}(0) = \delta_{ij}$. The fundamental problem treated by the transfer-matrix method is the evaluation of $A_{i,j}(n)$. The idea is to interpret $A_{i,j}(n)$ as an entry in a certain matrix. Define a $p \times p$ matrix $D = (D_{i,j})$ by

$$D_{i,j} = \sum_e w(e),$$

where the sum is over all edges e satisfying that its initial vertex is v_i and its final vertex is v_j . In other words, $D_{i,j} = A_{i,j}(1)$. The matrix D is called the *adjacency matrix* of \vec{G} , with respect to the weight function w .

Theorem 3.4. *Let $n \in \mathbb{N}$. Then the (i, j) -entry of D^n is equal to $A_{i,j}(n)$. Here we define $D^0 = I_p$ even if D is not invertible, where I_p is the identity matrix.*

Proof. See [41]. □

Rectangular lattice again

The transfer-matrix method gives us another way to compute $T_{L_{m,n}}(x, y)$ for a fixed width m at point (x, y) which is described in Calkin et. al. [17]. In this case we have the same restriction as before, a fixed width m for small values of m , but it has the advantage of being easily automatized.

Theorem 3.5 (Calkin et. al. 2003). *For indeterminates x and y and integers $n, m \geq 2$, m fixed, we have*

$$T_{L_{m,n}}(x+1, y+1) = x^{nm-1} X_m^t \cdot (\Lambda_m)^{n-1} \cdot \vec{1},$$

where X_m , a vector of length c_m , and Λ_m , a $c_m \times c_m$ matrix, depend on x, y and m but not n . And $\vec{1}$ is the vector of length c_m with all entries equal to 1.

The quantity c_m is the m -th Catalan number, so the method is just practical for small values of m . Computing the vectors X_m and the matrix Λ_m can be easily done in a computer.

For example, for $L_{2,n}$ we get that $T_{L_{2,n}}(x+1, y+1)$ equals

$$x^{2n-1}(x^{-1}, 1) \begin{pmatrix} x^{-1} + 3x^{-2} + yx^{-2} & 1 + 2x^{-1} \\ x^{-1} + 2x^{-2} + x^{-3} & 1 + 2x^{-1} + x^{-2} \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Wheels and whirls

The transfer matrix method takes a nice turn when combined with the physics idea of boundary conditions. In this case more lineal algebra is required but the method is still suitable to use in a computer algebra package.

A well-know family of self-dual graphs are wheel graphs, W_n . The graph W_n has $n+1$ vertices and $2n$ edges, see Figure 8. The rim of the wheel graph W_n is a circuit-hyperplane of the corresponding graphic matroid, the relaxation of this circuit-hyperplane gives the matroid W^n , the whirl matroid on n elements. In [19], S.-C. Chang and R. Shrock using results from [39] compute the Tutte polynomial of W_n .

$$\begin{aligned} T_{W_n}(x, y) &= \frac{1}{2^n} [(1+x+y) + ((1+x+y)^2 - 4xy)^{1/2}]^n \\ &\quad + \frac{1}{2^n} [(1+x+y) - ((1+x+y)^2 - 4xy)^{1/2}]^n + xy - x - y - 1. \end{aligned} \tag{36}$$

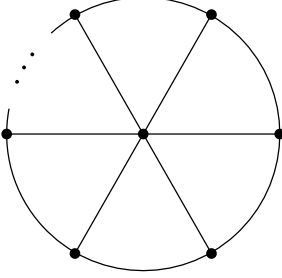


Figure 8: The n-wheel

From this expression is easy to compute an expression for the Tutte polynomial of whirls using equation (4).

$$\begin{aligned}
 T_{W_n}(x, y) &= \frac{1}{2^n} [(1 + x + y) + ((1 + x + y)^2 - 4xy)^{1/2}]^n \\
 &\quad + \frac{1}{2^n} [(1 + x + y) - ((1 + x + y)^2 - 4xy)^{1/2}]^n - 1.
 \end{aligned} \tag{37}$$

Now, the way S.-C. Chang and R. Shrock compute $T_{W_n}(x, y)$ is by using the Potts model partition function together with the transfer matrix method. Remarkably, the Tutte polynomial and the Potts model partition function are equivalent. But rather than defining the Potts model we invite the reader to check the surveys in [49, 4]. Here we show how to do the computation using the coboundary polynomial and the transfer method.

Let us compute the bad colouring polynomial for W_n when we have 3 colours. For this we define the 3×3 matrix D_3 .

$$\begin{pmatrix} t^2 & 1 & 1 \\ t & t & 1 \\ t & 1 & t \end{pmatrix}$$

The idea is that the entry ij of the matrix $(D_3)^n = D_3^n$ will contain all the contributions of bad edges to the bad colouring polynomial for all the colourings σ , with $\sigma(h) = 1$, $\sigma(1) = i$ and $\sigma(n) = j$ for the fan graph F_n , that is obtained from W_n by deleting one edge from the rim. Then, to get the bad-colouring polynomial of F_n we just add all the entries in D_3^n and multiply by $\lambda = 3$. To get the bad-colouring polynomial of W_n we take the trace of D_3^n and multiply by $\lambda = 3$. This works because, by taking the trace you are

considering just colourings with the same initial and final configuration, that is you are placing periodic boundary conditions.

Thus, for example, the trace in D_3^3 is $t^6 + 8t^3 + 6t^2 + 12t$, so $B_{W_3}(3, t) = 3t^6 + 24t^3 + 18t^2 + 36t$. By computing the eigenvalues of D_3 we obtain the bad colouring polynomial of W_n with $\lambda = 3$.

$$\begin{aligned} B_{W_n}(3, t) &= \frac{3}{2^n}[(t^2 + t + 1) + (t^4 - 2t^3 - t^2 + 10t + 1)^{1/2}]^n \\ &\quad + \frac{3}{2^n}[(t^2 + t + 1) + (t^4 - 2t^3 - t^2 + 10t + 1)^{1/2}]^n + 3(t - 1)^n. \end{aligned}$$

For arbitrary λ we need to compute the eigenvalues of the matrix of $\lambda \times \lambda$ given below

$$\begin{pmatrix} t^2 & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ t & 1 & t & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & 1 & 1 & \cdots & t \end{pmatrix}.$$

The eigenvalues of the matrix are

$$e_{1,2} = \frac{1}{2}[(t^2 + t + \lambda - 2) \pm (t^4 - 2t^3 - (2\lambda - 5)t^2 + (6\lambda - 8)t + (\lambda - 2)^2)^{1/2}],$$

each with multiplicity 1 and $e_3 = (t - 1)$ with multiplicity $\lambda - 2$. Thus the bad colouring polynomial of W_n is $\lambda(e_1^n + e_2^n + (\lambda - 2)e_3^n)$. To get formula (36) we just need to make the corresponding change of variables as seen at the beginning of the previous section.

Other examples that were computed by using this technique include Möbius strips, cycle strips and homogeneous clan graphs. For the corresponding formulas you can check [39, 18, 19, 20, 21].

3.3 Splitting the problem: 1, 2 and 3-sums

Our last technique is based on the recurrent idea of splitting a problem that otherwise may be too big. The notion of connectedness in matroid theory offers a natural setting for our purposes, but we do not explain this theory here and we direct the reader to the book of Oxley [37] for such a subject. Also, for 1, 2 and 3-sums of binary matroids the book of Truemper [43] is the best reference.

However, we would like to explain a little of the relation of 1, 2 and 3-sums and the splitting of a matroid. For a matroid $M = (E, r)$, a partition (X, Y) of E is an *exact k -separation*, for k a positive integer, if

$$\min\{|X|, |Y|\} \geq k$$

and

$$r(X) + r(Y) - r(M) = k - 1.$$

Now, a matroid M can be written as a 1-sum of two of its proper minors if and only if M has an exact 1-separation; and M can be written as a 2-sum of two of its proper minors if and only if M has an exact 2-separation. The situation is more complicated in the case of 3-sums and we just want to point out that if a *binary* matroid M has an exact 3-separation (X, Y) , with $|X|, |Y| \geq 4$, then there are binary matroids M_1 and M_2 such that $M = M_1 \oplus_3 M_2$.

1-sum

For n matroids M_1, M_2, \dots, M_n , on disjoint sets E_1, E_2, \dots, E_n the direct sum $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is the matroid (E, \mathcal{I}) where E is the union of the ground sets and $\mathcal{I} = \{I_1 \cup I_2 \cup \dots \cup I_n : I_i \in \mathcal{I}(M_i) \text{ for all } i \text{ in } \{1, 2, \dots, n\}\}$.

Directly from (1) it follows that the Tutte polynomial of the 1-sum is given by

$$T_{M_1 \oplus M_2 \oplus \dots \oplus M_n}(x, y) = \prod_{i=1}^n T_{M_i}(x, y).$$

2-sum

Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be matroids with $E_1 \cap E_2 = \{p\}$. If p is not a loop or an isthmus in M_1 or M_2 , then the 2-sum $M_1 \oplus_2 M_2$ of M_1 and M_2 is the matroid on $E_1 \cup E_2 \setminus \{p\}$ whose collection of independent sets is $\{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \text{ and either } I_1 \cup \{p\} \in \mathcal{I}_1 \text{ or } I_2 \cup \{p\} \in \mathcal{I}_2\}$

In [1], [12] and [37], we find recursive formulas for computing the Tutte polynomial of the matroid $M_1 \oplus_2 M_2$, in term of the matroids M_1 and M_2 . Here we present the formula given in [1] for $T_{M_1 \oplus_2 M_2}(x, y)$.

$$T_{M_1 \oplus_2 M_2} = \frac{1}{xy - x - y} \begin{bmatrix} T_{M_1/r} & T_{M_1 \setminus r} \end{bmatrix} \begin{bmatrix} x - 1 & -1 \\ -1 & y - 1 \end{bmatrix} \begin{bmatrix} T_{M_2/r} \\ T_{M_2 \setminus r} \end{bmatrix}, \quad (38)$$

where here we omit the variables x, y of each Tutte polynomial to avoid a cumbersome notation.

As an example, let us take R_6 that is the 2-sum of $U_{2,4}$ with itself. Observe that the election of the base point is irrelevant as its automorphism group is the symmetric group. The matroid $U_{2,4}/r \cong U_{1,3}$ and $U_{2,4} \setminus r \cong U_{2,3}$. Then, by using (38) we get that $T_{U_{2,4} \oplus_2 U_{2,4}}$ equals

$$\frac{1}{xy - x - y} \begin{bmatrix} y^2 + y + x & x^2 + x + y \end{bmatrix} \begin{bmatrix} x - 1 & -1 \\ -1 & y - 1 \end{bmatrix} \begin{bmatrix} y^2 + y + x \\ x^2 + x + y \end{bmatrix}.$$

Thus, we obtain that $T_{R_6}(x, y) = x^3 + 3x^2 + 4x + 2xy + 4y + 3y^2 + y^3$.

3-sum

The best known variant of a 3-sum of two matroids is called Δ -sum, see [43]. For 3-connected matroids M_1 and M_2 , the Δ -sum is usually denoted $M = M_1 \oplus_3^\Delta M_2$. When M_1 and M_2 are graphic matroids with corresponding graphs being G_1 and G_2 , the graph $G = G_1 \oplus_3^\Delta G_2$ is obtained by identifying a triangle of T_1 of G_1 with a triangle T_2 of G_2 into a triangle T , called the connector triangle. Finally, G is obtained by deleting the edges in the connector triangle.

Here, we present a formula to compute the Tutte polynomial of the Δ -sum of M_1 and M_2 , in terms of the Tutte polynomial of certain minors of the original matroids. The expression for $T_{M_1 \oplus_3^\Delta M_2}(x, y)$ was taken from [1] and its proof is based on the concept of bipointed matroids that is an extension of the pointed matroid introduced by Brylawski in [12]. Also, the work in [1] is more general as the author gives an expression for the Tutte polynomial of a certain type of general parallel connection of two matroids.

Let M_1 and M_2 be two matroids defined on E_1 and E_2 , respectively. Let T be equal to $E_1 \cap E_2 = \{p, s, q\}$, a 3-circuit and $N = M_1|T = M_2|T$. We require that in M_1 there exist circuits $U_1 \cup \{s\}$ with $U_1 \subseteq E_1 \setminus T$ and $U_2 \cup \{p\}$ with $U_2 \subseteq E_1 \setminus T$; similarly, we need that in M_2 there exist circuits $V_1 \cup \{s\}$ with $V_1 \subseteq E_2 \setminus T$ and $V_2 \cup \{p\}$ with $V_2 \subseteq E_2 \setminus T$.

For $i = 1, 2, \dots, 5$, we consider the following 5 minors Q_i of M_1 , on $E_1 \setminus T$, and 5 minors P_i of M_2 , on $E_2 \setminus T$.
 $Q_1 = M_1 \setminus p \setminus s \setminus q$, $Q_2 = M_1 \setminus p/s \setminus q$, $Q_3 = M_1/p \setminus s \setminus q$, $Q_4 = M_1/p/s/q$ and $Q_5 = M_1 \setminus p \setminus s/q$.

$P_1 = M_2 \setminus p \setminus s \setminus q$, $P_2 = M_2 \setminus p/s \setminus q$, $P_3 = M_2/p \setminus s \setminus q$, $P_4 = M_2/p/s/q$ and $P_5 = M_2 \setminus p \setminus s/q$.

We take the vectors over $\mathbb{Z}[x, y]$:

$$q^{\rightarrow} = [T_{Q_1}, T_{Q_2}, T_{Q_3}, T_{Q_4}, T_{Q_5}],$$

$$p^{\rightarrow} = [T_{P_1}, T_{P_2}, T_{P_3}, T_{P_4}, T_{P_5}],$$

again here we omit the variables x, y of each Tutte polynomial to avoid a cumbersome notation.

Finally, the formula for the 3-sum (Δ -sum) of M_1 and M_2 , it is as follows.

$$T_{M_1 \oplus_3^\Delta M_2}(x, y) = (q^{\rightarrow})^t \left[\frac{1}{-1 - x - y + xy} \mathbf{C} \right] p^{\rightarrow} \quad (39)$$

where the matrix \mathbf{C} is given by:

$$\begin{bmatrix} \frac{(1-y)^2}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{2}{-x-y+xy} & \frac{1-y}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & 1 & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & 1 & 1 & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{2}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{(1-x)^2}{-x-y+xy} & \frac{1-x}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & 1 \end{bmatrix}$$

As an example let us take F_8 that is the 3-sum of F_7 and F_7^- along a 3-circuit. In this case we have for F_7 the following table, where in the first column we present the 5 minors we need, then the second column has the corresponding matroid, and the third column has the corresponding Tutte polynomial of that matroid.

Minor	Matroid	polynomial
$F_7 \setminus p \setminus s \setminus q$	$U_{3,4}$	$x^3 + x^2 + x + y$
$F_7 \setminus p/s \setminus q$	$U_{1,2} \oplus U_{1,2}$	$(x + y)^2$
$F_7/p \setminus s \setminus q$	$U_{1,2} \oplus U_{1,2}$	$(x + y)^2$
$F_7/p/s/q$	$U_{1,4}$	$y^3 + y^2 + y + x$
$F_7 \setminus p \setminus s/q$	$U_{1,2} \oplus U_{1,2}$	$(x + y)^2$

Similarly for F_7^- we have the corresponding table:

Minor	Matroid	polynomial
$F_7^- \setminus p \setminus s \setminus q$	$U_{3,4}$	$x^3 + x^2 + x + y$
$F_7^- \setminus p/s \setminus q$	C_3 plus a paralell edge	$x^2 + x + xy + y + y^2$
$F_7^-/p \setminus s \setminus q$	$U_{1,2} \oplus U_{1,2}$	$(x + y)^2$
$F_7^-/p/s/q$	$U_{1,4}$	$y^3 + y^2 + y + x$
$F_7^- \setminus p \setminus s/q$	$U_{1,2} \oplus U_{1,2}$	$(x + y)^2$

By using (39) with the values of the above tables we get $T_{F_8}(x, y) = x^4 + 4x^3 + 10x^2 + 8x + 12xy + 8y + 10y^2 + 4y^3 + y^4$.

There is a general concept of k – *sum* for matroids and a formula exists for this general notion, however the formula is quit intricate, so we refer the reader to the original paper of Bonin and de Mier in [8].

Thickening, stretch and tensor product

Given a matroid M and a positive integer k , the matroid $M^{(k)}$ is the matroid obtained from M by replacing each non-loop element by k parallel

elements and replacing each loop by k loops. The matroid $M^{(k)}$ is called the k -thickening of M . In [16] the following formula is given for the Tutte polynomial of $M^{(k)}$ in terms of that of M .

$$T_{M^{(k)}}(x, y) = (y^{k-1} + y^{k-2} + \dots + y + 1)^{r(M)} T_M\left(\frac{y^{k-1} + y^{k-2} + \dots + y + x}{y^{k-1} + y^{k-2} + \dots + y + 1}, y^k\right). \quad (40)$$

A proof by using the recipe theorem can be read in the aforementioned reference. Here we hint a simple proof by noticing that any flat of $M^{(k)}$ is the k -thickening of a flat of M . Thus, by equation (20)

$$\bar{\chi} M^{(k)}(q, t) = \bar{\chi} M(q, t^k).$$

And thus, by using equations (23) and (22) we obtained the formula.

The dual operation to k -thickening is that of k -stretch that is defined similarly. The matroid $M_{(k)}$ is the matroid obtained by replacing each non-isthmus of M by k elements in series and replacing each isthmus by k isthmuses. The matroid $M_{(k)}$ is called the k -stretch. It is not difficult to prove that $M_{(k)} \cong ((M^*)^{(k)})^*$ and so, we obtained the corresponding formula for $T_{M_{(k)}}$.

$$T_{M_{(k)}}(x, y) = (x^{k-1} + x^{k-2} + \dots + x + 1)^{r(M^*)} T_M\left(x^k, \frac{x^{k-1} + x^{k-2} + \dots + x + y}{x^{k-1} + x^{k-2} + \dots + x + 1}\right). \quad (41)$$

More generally we have the following operation called the *tensor product*. A *pointed* matroid N_d is a matroid on a ground set which includes a distinguished element, the point d , which will be assumed to be neither a loop or coloop. For an arbitrary matroid M and a pointed matroid N_d , the *tensor product* $M \otimes N_d$ is the matroid obtained by taking a 2-sum of M with N_d at each point e of M and the distinguished point d of N_d . The Tutte polynomial of $M \otimes N_d$, where $M = (E, r)$ is then given by

$$T_{M \otimes N_d}(x, y) = f^{|E|-r(E)} g^{r(E)} T_M\left(\frac{(x-1)f+g}{g}, \frac{f+(y-1)g}{f}\right), \quad (42)$$

where $f = f(x, y)$ and $g = g(x, y)$ are polynomials which are determined by the equations

$$\begin{aligned} (x-1)f(x, y) + g(x, y) &= T_{N_d \setminus d}(x, y) \\ f(x, y) + (y-1)g(x, y) &= T_{N_d/d}(x, y). \end{aligned}$$

The proof of the formula uses a generalization of the recipe theorem to pointed matroids and can be found in [15], here we follow the exposition in [16, 27]. Observe that when a matroid N has a transitive automorphism group the choice of the distinguished point d is immaterial. Thus, if N is $U_{k,k+1}$, $k \geq 1$, we get the k -stretch and if N is $U_{1,k+1}$, $k \geq 1$, we get the k -thickening.

4 Application: Small matroids

Let us put these techniques in practice and compute some Tutte polynomials for matroids with a small number of elements, these are matroids from the appendix in Oxley's book [37]. We will try, whenever possible, to check the result by using two techniques. Soon, the reader will realize that a fair amount of the matroids considered are sparse paving and that computing the Tutte polynomial for them is quite easy.

Matroid $U_{2,4}$

The Tutte polynomial of the uniform matroid $U_{2,4}$ can be computed using equation (3).

$$T_{U_{2,4}}(x, y) = x^2 + 2x + 2y + y^2. \quad (43)$$

Also this matroid is the 2-whirl W^2 so you can check this result using equation (37)

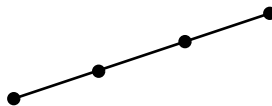


Figure 9: The uniform matroid $U_{2,4}$

Matroids W_3 , W_4 , W^3 and W^4

The 3-wheel $M(W_3)$, which is isomorphic to the graphic matroid $M(K_4)$, is a sparse paving matroid with $\lambda = 4$ circuit-hyperplanes, so by using (5) we get

$$T_{W_3}(x, y) = x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3. \quad (44)$$

Of course, the above polynomial can be checked using (36). The only relaxation of $M(W_3)$ is the 3-whirl W^3 , so by using (4) we obtain its Tutte polynomial. This can be checked by using (37).

$$T_{W^3}(x, y) = x^3 + 3x^2 + 3x + 3xy + 3y + 3y^2 + y^3. \quad (45)$$

The 4-wheel, $M(W_4)$, and 4-whirl, W^4 , are matroids whose geometric representation are shown in Figures 10 and 11. Their Tutte polynomial can be computed using (36) and (37).

$$\begin{aligned} T_{W_4}(x, y) &= \frac{(1 + x + y + \sqrt{(1 + x + y)^2 - 4xy})^4}{16} + \\ &\quad \frac{(1 + x + y - \sqrt{(1 + x + y)^2 - 4xy})^4}{16} + xy - x - y - 1 \\ &= x^4 + 4x^3 + 6x^2 + 3x + 4x^2y + 4xy^2 + 9xy + 3y + 6y^2 + \\ &\quad 4y^3 + y^4. \end{aligned}$$

$$T_{W^4}(x, y) = x^4 + 4x^3 + 6x^2 + 4x + 4x^2y + 4xy^2 + 8xy + 4y + 6y^2 + 4y^3 + y^4. \quad (46)$$

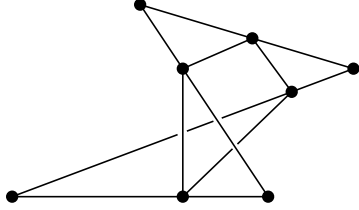


Figure 10: The 4-wheel

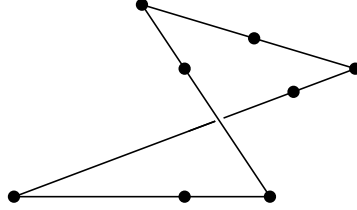


Figure 11: The 4-whirl

Matroids Q_6 , P_6 , R_6 and $U_{3,6}$

We use (4) to compute the Tutte polynomial of Q_6 , P_6 and $U_{3,6}$, see Figure 12 for a geometric representation of these matroids. The matroid $U_{3,6}$ is uniform, from (3) we get

$$T_{U_{3,6}}(x, y) = x^3 + 3x^2 + 6x + 6y + 3y^2 + y^3.$$

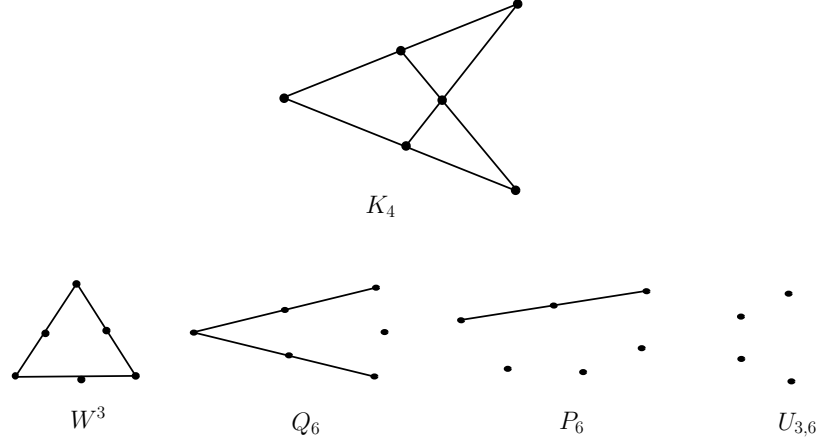


Figure 12: $M(K_4)$, W^3 , Q_6 , P_6 and $U_{3,6}$

As $U_{3,6}$ is the only relaxation of P_6 , from (4) its Tutte polynomial is

$$T_{P_6}(x, y) = x^3 + 3x^2 + 5x + xy + 5y + 3y^2 + y^3.$$

Similarly, P_6 is the only relaxation of Q_6 . Thus, from the above equation and (4) we obtain

$$T_{Q_6}(x, y) = x^3 + 3x^2 + 4x + 2xy + 4y + 3y^2 + y^3. \quad (47)$$

It is worth mentioning that P_6 is a relaxation of R_6 , see Figure 13. Thus, Q_6 and R_6 have the same Tutte polynomial. This is not at all uncommon, see [37, 7]. Also, R_6 is sparse paving so by (5) its Tutte polynomial is

$$\begin{aligned} T_{R_6}(x, y) &= (x-1)^3 + 6(x-1)^2 + 13x - 10 + 2xy + 13y + 6(y-1)^2 \\ &\quad + (y-1)^3 \\ &= x^3 + 3x^2 + 4x + 2xy + 4y + 3y^2 + y^3. \end{aligned}$$

Matroids F_7 , F_7^- and their duals

One of the most mentioned matroid in the literature is the Fano matroid, F_7 , also known as the projective plane $PG(2, 2)$ or the unique Steiner system $S(2, 3, 7)$. Being a Steiner triple system implies that the Fano matroid is

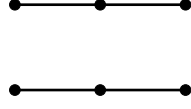


Figure 13: The matroid R_6

sparse paving, see [50]. As it has 7 circuit-hyperplanes we obtain from (5) that the Tutte polynomial of F_7 is

$$\begin{aligned} T_{F_7}(x, y) &= (x-1)^3 + 7(x-1)^2 + 14x - 21 + 7xy + 28y + 21(y-1)^2 + \\ &\quad 7(y-1)^3 + (y-1)^4 \\ &= x^3 + 4x^2 + 3x + 7xy + 3y + 6y^2 + 3y^3 + y^4. \end{aligned} \quad (48)$$

The matroid F_7 is representable over any field of characteristic 2, so the above calculation can be checked using the program in [2] with the matrix

$$\left[\begin{array}{c|cccc} & 1 & 1 & 0 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 \end{array} \right].$$

By using (2) the Tutte polynomial of F_7^* is

$$T_{F_7^*}(x, y) = x^4 + 3x^3 + 6x^2 + 3x + 7xy + 3y + 4y^2 + y^3. \quad (49)$$

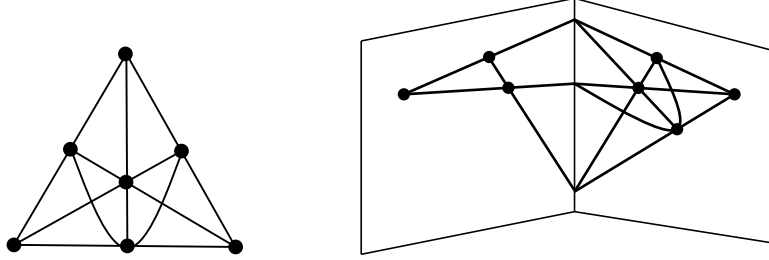


Figure 14: The Fano matroid and its dual

The matroids F_7^- and $(F_7^-)^*$ are the corresponding relaxations of F_7 y F_7^* , thus we get that

$$T_{F_7^-}(x, y) = x^3 + 4x^2 + 4x + 6xy + 4y + 6y^2 + 3y^3 + y^4,$$

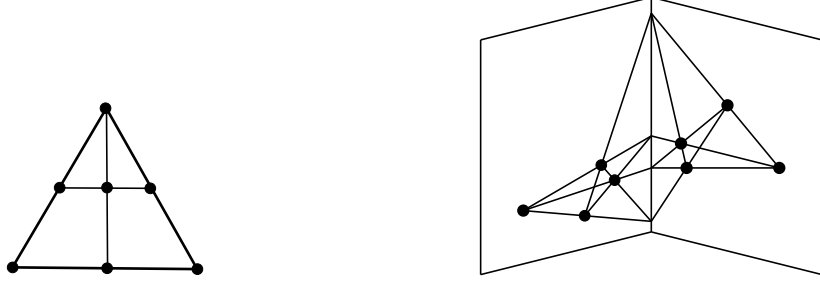


Figure 15: Geometric representation of P_7 and P_8

and

$$T_{(F_7^-)^*}(x, y) = x^4 + 3x^3 + 6x^2 + 4x + 6xy + 4y + 4y^2 + y^3.$$

Matroids P_7, P_8 and Q_3

The matroid P_7 is a rank-3 sparse paving matroid, thus by using (5) we get its Tutte polynomial,

$$T_{P_7}(x, y) = x^3 + 4x^2 + 5x + 5xy + 5y + 6y^2 + 3y^3 + y^4.$$

The matrix that represents P_7 over a field different from $GF(2)$ is

$$\left[\begin{array}{c|cccc} & 1 & 0 & 1 & 1 \\ I_3 & 1 & 1 & 0 & 1 \\ & a & 1 & a-1 & 0 \end{array} \right]$$

with $a \notin \{0, 1\}$. Taking $a = 2$ we have a representation of P_7 over $GF(3)$, see [37]. Thus, the above calculation of the Tutte polynomial can be checked using the computer program in [2].

The matroid P_8 is also sparse paving and its representation over $GF(3)$ is as below,

$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & -1 \\ & 1 & 0 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 \\ & -1 & 1 & 1 & 0 \end{array} \right].$$

Thus, its Tutte polynomial can be computed using either (5) or the program in [2], and you get as a result the following polynomial.

$$T_{P_8}(x, y) = x^4 + 4x^3 + 10x^2 + 10x + 10xy + 10y + 10y^2 + 4y^3 + y^4.$$

The rank-3 ternary Dowling geometry Q_3 has geometric representation shown in Figure 16.

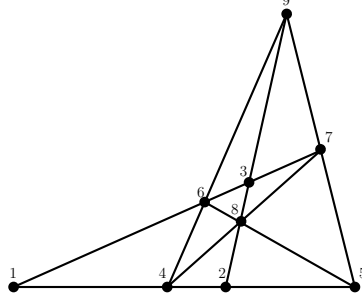


Figure 16: Geometric representation of Q_3

The matroid Q_3 is representable over a field F if and only if the characteristic of F is different from 2. A matrix that represents Q_3 over $GF(3)$ is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right].$$

Thus, we can compute its Tutte polynomial using [2] and obtain

$$T_{Q_3}(x, y) = x^3 + 6x^2 + 8x + 3xy^2 + 10xy + 8y + 12y^2 + 10y^3 + 6y^4 + 3y^5 + y^6.$$

Note, however, that this matroid is paving, so we could have used Proposition 2.8 to get the same result.

Matroid W_+^3

The matroid W_+^3 is obtained from W^3 by adding an element in parallel and so, the matroid is not paving. In this case, we can use Definition 2.3 to compute the Tutte polynomial. In Figure 17 we show how we are using deletion and contraction to find matroids where the Tutte polynomial is either known or

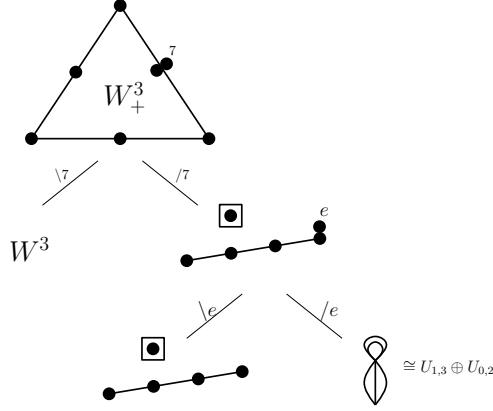


Figure 17: Deletion and contraction reduction for W_+^3

easy to compute.

$$\begin{aligned}
T_{W_+^3}(x, y) &= T_{W^3}(x, y) + y \cdot (T_{U_{2,4}}(x, y)) + T_{U_{1,3} \oplus U_{0,2}}(x, y) \\
&= (x^3 + 3x^2 + 3xy + 3x + 3y + 3y^2 + y^3) + y \cdot (x^2 + 2x + 2y + y^2) \\
&\quad + (y^4 + y^3 + xy^2) \\
&= x^3 + 3x^2 + 3x + x^2y + 5xy + xy^2 + 3y + 5y^2 + 3y^3 + y^4.
\end{aligned}$$

Matroids $AG(3, 2)$, $AG(3, 2)'$, R_8 , Q_8 , F_8 and L_8

The second affine plane that we consider here is $AG(3, 2)$, Figures 18 and 19 show two ways of representing the matroid. For example, in Figure 19 the planes with 4 points are the six faces of the cube, the six diagonal planes $\{1, 2, 7, 8\}$, $\{2, 3, 5, 8\}$, $\{3, 4, 5, 6\}$, $\{1, 4, 6, 7\}$, $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$, plus the two twisted planes $\{1, 3, 6, 8\}$ and $\{2, 4, 5, 7\}$. Each of these planes is a circuit-hyperplane. It is not difficult to check that $AG(3, 2)$ is isomorphic to the unique Steiner system $S(3, 4, 8)$, thus, as explained for $AG(2, 3)$, it is sparse paving and its Tutte polynomial is

$$T_{AG(3,2)}(x, y) = x^4 + 4x^3 + 10x^2 + 6x + 14xy + 6y + 10y^2 + 4y^3 + y^4. \quad (50)$$

By using (4), we can compute the Tutte polynomial of $AG(3, 2)'$, the unique relaxation of $AG(3, 2)$ that here we obtained by relaxing the twisted

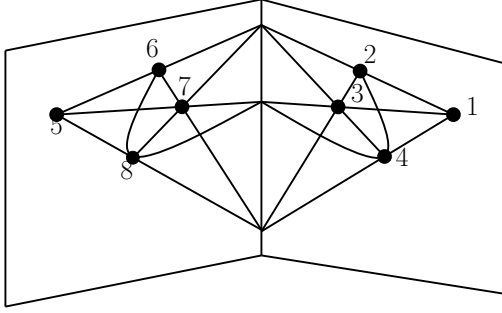


Figure 18: The affine plane $AG(3, 2)$

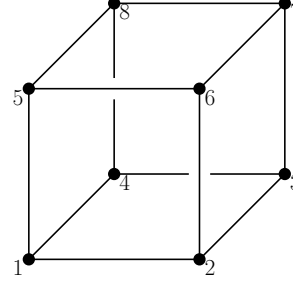


Figure 19: The cube with 8 points

plane $\{2, 4, 5, 7\}$. Thus, the resulting polynomial is

$$T_{AG(3,2)'}(x, y) = x^4 + 4x^3 + 10x^2 + 7x + 13xy + 7y + 10y^2 + 4y^3 + y^4. \quad (51)$$

Now, $AG(3, 2)'$ has two relaxations, R_8 and F_8 . The matroid R_8 is obtained by relaxing from $AG(3, 2)'$ the other twisted plane $\{1, 3, 6, 8\}$. On the other hand, F_8 is obtained from $AG(3, 2)'$ by relaxing a diagonal plane. The geometric representation of F_8 is shown in Figure 20.

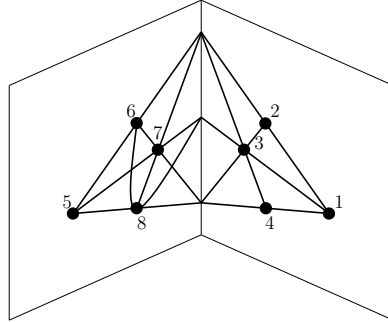


Figure 20: The matroid F_8

The matroid R_8 is representable over any field except for $GF(2)$ while F_8 is not representable; however, they have the same Tutte polynomial as both are relaxations of the same matroid.

$$T_{R_8}(x, y) = T_{F_8}(x, y) = x^4 + 4x^3 + 10x^2 + 8x + 12xy + 8y + 10y^2 + 4y^3 + y^4.$$

The unique relaxation of R_8 is Q_8 , and is obtained by relaxing one of the six diagonal planes of R_8 . From the previous equation and (4) we get

$$T_{Q_8}(x, y) = x^4 + 4x^3 + 10x^2 + 7x + 11xy + 7y + 10y^2 + 4y^3 + y^4.$$

To end this subsection we consider the matroid L_8 which is a rank-4 sparse paving matroid with 8 elements and its circuit-hyperplanes are the six faces of the cube plus the two twisted planes $\{1, 8, 3, 6\}$ and $\{2, 7, 4, 5\}$, see Figure 19. Thus, its Tutte polynomial is

$$\begin{aligned} T_{L_8}(x, y) &= (x-1)^4 + 8(x-1)^3 + 28(x-1)^2 + 48x - 42 + 8xy + 48y + \\ &\quad 28(y-1)^2 + 8(y-1)^3 + (y-1)^4 \\ &= x^4 + 4x^3 + 10x^2 + 12x + 8xy + 12y + 10y^2 + 4y^3 + y^4. \end{aligned}$$

Matroids S_8, T_8 and J

The matroid S_8 has a geometric representation shown in Figure 21 together with its representation over $GF(2)$. Its Tutte polynomial can be computed using the program in [2].

$$T_{S_8}(x, y) = x^4 + 4x^3 + 7x^2 + 4x + 10xy + 3xy^2 + 3x^2y + 4y + 7y^2 + 4y^3 + y^4. \quad (52)$$

Note that the matroid is self-dual but is not paving as it has a 3-circuit. However, we can check the above computation using deletion and contraction. When we contract the forth column in the representation we obtain the representation of F_7 , and we have already computed the Tutte polynomial of this matroid. Now, when we delete the same element, we obtain a rank 4 graphic matroid. The graph is $K_{2,4}$ with an edge contracted. If we call this graph H , by using (7) we obtain $T_H = T_{K_{2,4}} - yT_{K_{2,3}}$. These polynomial can be computed either using the general method for complete bipartite graphs given in (30) or by using the formula for computing the Tutte polynomial of the 2-stretching of the graphs with two and three parallel edges respectively, given in (41). In both cases, we get

$$T_H = x^4 + 3x^3 + 3x^2 + x + 3xy + 3x^2y + 3xy^2 + y + y^2 + y^3. \quad (53)$$

By adding (53) and (48) we get (52)

The matroid T_8 is representable over a field \mathbb{F} if and only if the characteristic is 3. We show its geometric representation in Figure 22. A representation

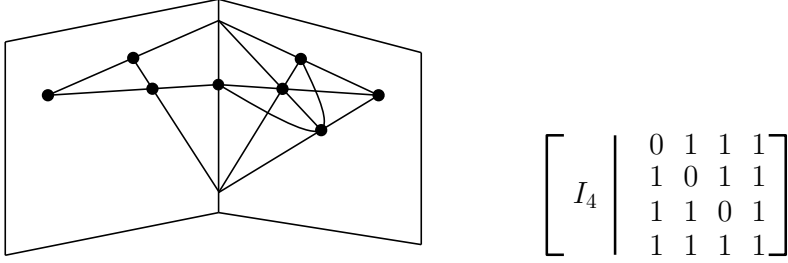


Figure 21: On the left hand side we give the geometric representation of the matroid S_8 and on the right hand side the matrix that represents it over $GF(2)$.

of T_8 over $GF(3)$ is $[I_4|J_4 - I_4]$, where J_4 is the matrix of 1's. This matroid is self-dual and sparse paving so its Tutte polynomial is

$$T_{T_8}(x, y) = x^4 + 4x^3 + 10x^2 + 9x + 11xy + 9y + 10y^2 + 4y^3 + y^4.$$

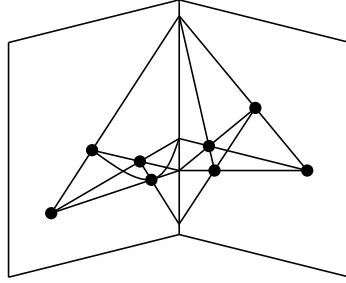


Figure 22: The matroid T_8

In Figure 23 we show the geometric representation of the matroid J . It is a self-dual matroid that is not paving, as it has a 3-circuit, and its representation over $GF(3)$ is

$$\left[\begin{array}{c|cccc} I_4 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 \end{array} \right]$$

where the labelling of the columns correspond to the labelling of the elements in the geometric representation of Figure 23. The Tutte polynomial can be computed using [2].

$$T_J(x, y) = x^4 + 4x^3 + 7x^2 + 6x + 3x^2y + 3xy^2 + 8xy + 6y + 7y^2 + 4y^3 + y^4.$$

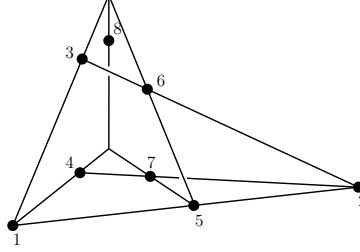


Figure 23: The matroid J

This computation can be checked by, for example, contracting the element labelled 2 to obtain the graphic matroid of the graph that is the 2-stretching of C_4 minus an edge. Now, the matroid $J \setminus 2$ is sparse paving and contains 5 circuit-hyperplanes. By adding the Tutte polynomial of these two matroids you get the same result as above.

The Vámos matroid

The 8-element rank-4 matroid whose geometric representation is shown in Figure 24 is known as the *Vámos* matroid and it is usually denoted by V_8 . It occurs quite frequently in Oxley's book and has many interesting properties, for example is not representable over any field and it is sparse paving, see [37]. A related matroid V_8^+ has the same ground set and is defined in the same way as V_8 but with $\{5, 6, 7, 8\}$ added as a hyperplane. In fact, V_8 is obtained from V_8^+ by relaxing $\{5, 6, 7, 8\}$. Notice that both are self-dual. Thus the Tutte polynomial of these two matroids are

$$\begin{aligned} T_{V_8}(x, y) &= (x-1)^4 + 8(x-1)^3 + \binom{8}{2}(x-1)^2 + \binom{8}{3}(x-1) \\ &\quad + \binom{8}{4} + 5(xy-x-y) + \binom{8}{5}(y-1) + \binom{8}{6}(y-1)^2 \\ &\quad + 8(y-1)^3 + (y-1)^4 \\ &= x^4 + 4x^3 + 10x^2 + 15x + 5xy + 15y + 10y^2 + 4y^3 + y^4 \end{aligned} \quad (54)$$

and

$$T_{V_8^+}(x, y) = x^4 + 4x^3 + 10x^2 + 14x + 6xy + 14y + 10y^2 + 4y^3 + y^4. \quad (55)$$

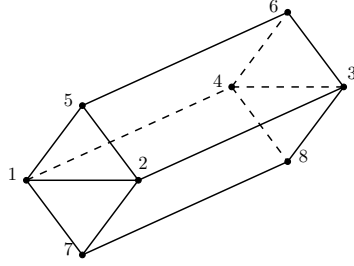


Figure 24: Geometric representation of V_8

Matroids R_8, R_9, R_{10} and R_{12}

The real affine cube, R_8 , is represented over all fields of characteristic other than two by the matrix

$$\left[\begin{array}{c|cccc} I_4 & -1 & 1 & 1 & 1 \\ & 1 & -1 & 1 & 1 \\ & 1 & 1 & -1 & 1 \\ & 1 & 1 & 1 & -1 \end{array} \right].$$

The matroid is sparse paving and its Tutte polynomial is

$$T_{R_8}(x, y) = x^4 + 4x^3 + 10x^2 + 8x + 12xy + 8y + 10y^2 + 4y^3 + y^4. \quad (56)$$

On the contrary, the ternary Reid geometry, R_9 , is not paving but it is representable if and only if the characteristic of the field is three. The matrix that represents R_9 over $GF(3)$ is

$$\left[\begin{array}{c|cccccc} I_3 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & -1 & -1 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 1 & -1 \end{array} \right]$$

and its Tutte polynomial can be computed using the program in [2].

$$T_{R_9}(x, y) = x^3 + 6x^2 + 8x + 11xy + 2xy^2 + 8y + 13y^2 + 10y^3 + 6y^4 + 3y^5 + y^6.$$

This was computed before in section 2.3 with the same result.

The unique 10-element regular matroid that is neither graphic or co-graphic, R_{10} , has the property that any every single-element deletion is isomorphic to $M(K_{3,3})$, and every single-element contraction is isomorphic to $M^*(K_{3,3})$, then by using the polynomial in (31) and (2) we obtain

$$T_{R_{10}}(x, y) = x^5 + 5x^4 + 15x^3 + 20x^2 + 10x + 15x^2y + 30xy + 15xy^2 + 10y + 20y^2 + 15y^3 + 5y^4 + y^5. \quad (57)$$

Another important regular matroid that is neither graphic or cographic is R_{12} , which has a matrix representation over $GF(2)$ given by

$$\left[\begin{array}{c|cccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ I_6 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right].$$

As this matroid is not paving we use the program in [2] to compute its Tutte polynomial.

$$T_{R_{12}}(x, y) = x^6 + 6x^5 + 19x^4 + 35x^3 + 35x^2 + 14x + 2x^4y + 19x^3y + 53x^2y + 17x^2y^2 + 56xy + 53xy^2 + 19xy^3 + 2xy^4 + 14y + 35y^2 + 35y^3 + 19y^4 + 6y^5 + y^6.$$

Pappus and non-Pappus matroids

The geometric representation of the *Pappus* matroid is shown in Figure 25. From the picture it is clear that any two points are in a unique line and that each line is a circuit-hyperplane in a rank-3 matroid. We conclude that the matroid is sparse paving with $\lambda = 9$ circuit-hyperplanes and its Tutte polynomial is

$$\begin{aligned} T_M(x, y) &= (x-1)^3 + 9(x-1)^2 + 27x - 78 + 9xy + 117y + 126(y-1)^2 + \\ &\quad 84(y-1)^3 + 36(y-1)^4 + 9(y-1)^5 + (y-1)^6 \\ &= x^3 + 6x^2 + 12x + 9xy + 12y + 15y^2 + 10y^3 + 6y^4 + 3y^5 + y^6. \end{aligned}$$

The *non-Pappus* matroid is a relaxation of the Pappus matroid and its geometric representation is shown in Figure 26. From the previous equation and (4) we obtained

$$\begin{aligned} T_M(x, y) &= x^3 + 6x^2 + 12x + 9xy + 12y + 15y^2 + 10y^3 + 6y^4 + 3y^5 + y^6 \\ &\quad - xy + x + y \\ &= x^3 + 6x^2 + 13x + 8xy + 13y + 15y^2 + 10y^3 + 6y^4 + 3y^5 + y^6. \end{aligned}$$

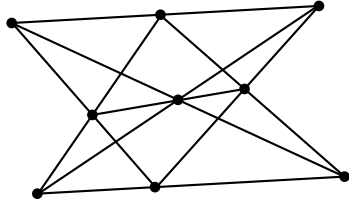


Figure 25: Geometric representation of the Pappus matroid

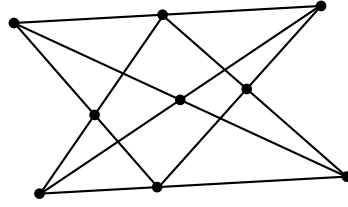


Figure 26: Geometric representation of the non-Pappus matroid

Matroid non-Desargues

The *non-Desargues* has the geometric representation shown in Figure 27. It is a rank-3 matroid with 10 elements that is sparse paving with the 9 circuit-hyperplanes that are the 9 lines in the picture. Its Tutte polynomial is

$$T_M(x, y) = x^3 + 7x^2 + 19x + 9xy + 19y + 21y^2 + 15y^3 + 10y^4 + 6y^5 + 3y^6 + y^7. \quad (58)$$

The Steiner systems $S(2, 3, 13)$ and $S(5, 6, 12)$

Finally, we consider two Steiner systems from the Appendix of Oxley's book. In general any Steiner system $S(v - 1, v, n)$ is sparse paving, see [50]. Thus, the Steiner triple system $S(2, 3, 13)$ is a rank-3 sparse paving matroid with 13 elements and $\lambda = 26$ circuit-hyperplanes, then its Tutte polynomial is

$$\begin{aligned} T_{S(2,3,13)}(x, y) &= x^3 + 10x^2 + 29x + 26xy + 29y + 45y^2 + 36y^3 \\ &\quad + 28y^4 + 21y^5 + 15y^6 + 10y^7 + 6y^8 + 3y^9 + y^{10}. \end{aligned} \quad (59)$$

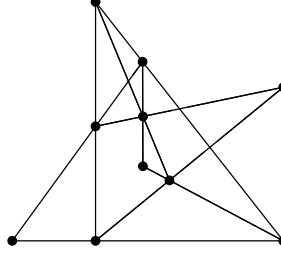


Figure 27: Geometric representation of the non-Desargues matroid

The Steiner system $S(5, 6, 12)$ has rank 6, 12 elements and 132 circuit-hyperplanes. The Tutte polynomial is, by using equation (5),

$$\begin{aligned} T_{S(5,6,12)}(x, y) = & x^6 + 6x^5 + 21x^4 + 56x^3 + 126x^2 + 120x + 132xy \\ & + 120y + 126y^2 + 56y^3 + 21y^4 + 6y^5 + y^6. \end{aligned} \quad (60)$$

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